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Beliefs on Individual Variables from a Single Source to Beliefs on the Joint Space under Dempster-Shafer Theory: An Algorithm

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Abstract: It is quite common in real world situations to form beliefs under Dempster-Shafer (DS) theory on various variables from a single source. This is true, in particular, in auditing. Also, the judgment about these beliefs is easily made in terms of simple support functions on individual variables. However, for propagating beliefs in a network of variables, one needs to convert these beliefs on individual variables to beliefs on the joint space of the variables pertaining to the single source of evidence. Although there are many possible solutions to the above problem that will yield beliefs on the joint space with the desired marginal beliefs, there is no method that will guarantee that the beliefs are derived from the same source, fully dependent evidence. In this article, we describe such a procedure based on a maximal order decomposition algorithm. The procedure is computationally efficient and is supported by objective chi-square and entropy criteria. While such assignments are not unique, alternative procedures that have been suggested, such as linear programming, are more computationally intensive and result in similar m-value determinations. It should be noted that our maximal order decomposition (i.e., minimum entropy) approach provides m-values on the joint space for fully dependent items of evidence.

1 INTRODUCTION

It is quite common, especially in auditing, to use one source of evidence to form beliefs under Dempster-Shafer theory (Shafer 1976, Srivastava & Mock 2002) on two or more variables in a decision. For example, in an audit of the financial statements, the auditor performs a test of confirmation of accounts receivables where he/she sends letters to a given number of randomly selected customers of the company being audited asking whether they owe the specified amount of money to the company. Such a confirmation provides support to two assertions, 'Existence' and 'Valuation'. Valuation implies that the account balance is correctly stated and Existence implies that the customer really exists, i.e., the customer is not fictitious. The level of support or belief that the account is valued correctly, in general, would differ from the level of belief that the customer does really exist. These beliefs can easily be expressed in terms of simple support functions on each variable, ‘Existence’ and ‘Valuation’. However, for the purpose of propagating beliefs in a network (Shenoy and Shafer 1990) one needs to convert these beliefs into a belief function on the joint space of the variables pertaining to the single source of evidence. This paper deals with such a conversion algorithm.

The main purpose of this article is to describe an algorithm that converts beliefs in terms of m-values, the *basic probability assignment function* (Shafer 1976), that are defined on individual variables but have come from the same source of evidence to m-values on the joint space of the variables. Such a conversion is needed in order to propagate beliefs in a network of variables and to preserve the interdependencies among the items of evidence. In auditing, it is quite common to use one source of evidence to form beliefs on different variables. Before we describe an example of the above situation, we want to give a brief introduction to the audit process below and show how important the above issue is for the auditor.

The accounting profession defines auditing as (see, e.g., Arens, Elder, and Beasley 2006):

“Auditing is the accumulation and evaluation of evidence to determine and report on the degree of correspondence between the information and established criteria (p. 4).”

There are three important steps in the above definition that one should make a note of. The first step, of course, is the accumulation of evidence. The second step is the evaluation of evidence in terms of the degree of correspondence between the information and established criteria. The third step deals with the aggregation of all the evidence to form an opinion whether the information of the entity is in accordance with the established criteria. For the audit of financial statements (FS),[1] the information consists of the account balances reported on the FS and the established criteria are the Generally Accepted Accounting Principles (GAAP). Examples of accounts on the balance sheet would be cash, accounts receivable, inventory, etc., and on the income statement would be sales, cost of goods sold, expenses, etc.

In essence, the auditor accumulates sufficient evidence related to the financial statements to express an opinion that the financial statements present fairly the financial position of the company in accordance with GAAP. The question is what is fairly? It is assumed that the FS are the representations of management of the company. When a company issues its FS, the management is making certain assertions about the numbers reported in the FS. These assertions are called management assertions. The American Institute of Certified Public Accountants through the Statement on Auditing Standards No. 31 (AICPA 1980, see also SAS 106, AICPA 2006) classifies these assertions into five categories: ‘Existence or Occurrence’, ‘Completeness’, ‘Rights and Obligation’, ‘Valuation or Allocation’, ‘Presentation and Disclosure’. It is assumed that when all the assertions related to an account are met then the account is fairly stated.

In order to facilitate accumulation of evidence to determine whether each management assertion is met, the AICPA has developed its own nine objectives called audit objectives: Existence, Completeness, Accuracy, Classification, Cutoff, Detail Tie-in, Realizable value, Rights and Obligations, Presentation and Disclosure (Arens, Elder, and Beasley, 2006, p. 150). These objectives are closely related to the management assertions. For example, audit objectives: Existence, Completeness, and Rights and Obligations, respectively, correspond to management assertions: Existence or Occurrence, Completeness, and Rights and Obligation. The audit objectives: Accuracy, Classification, Cutoff, Detail Tie-in, and Realizable value relate to ‘Valuation and Allocation’ assertion because they all deal with the valuation of the account balance on the FS. The audit objective ‘Presentation and Disclosure’ relates to the management assertion ‘Presentation and Disclosure’.

Thus, in an audit, the auditor collects enough evidence to make reasonably sure that each assertion of an account is met and consequently each account is fairly stated and finally making a decision on the fair presentation of the whole FS. There are two important points related to the above decision process. One deals with the nature of uncertainties associated with the audit evidence and the other deals with the structure. Srivastava and Shafer (1992) have argued that belief functions provide a better framework for representing uncertainties associated with the audit evidence than probability theory (see also, Akresh , Loebbecke, and Scott 1988, Harrison, Srivastava, and Plumlee 2002, Srivastava 1993, Shafer and Srivastava 1990). Regarding the structure of evidence, it is well known that it forms a network of variables; variable being the accounts on the FS, the audit objectives of the accounts, and the FS as a whole (see, e.g., Srivastava 1995, Srivastava, Dutta and Johns 1996, Srivastava and Lu 2002). Thus, the process of aggregating all the audit evidence to form an opinion is essentially the process of propagating beliefs in a network of variables (Shenoy and Shafer 1990, Srivastava 1995).

The network structure arises because one item of evidence bears on more than one variable in the network. For example, confirmations of receivables[2] bear on the following two audit objectives of the account: 'Existence' and 'Valuation'. The auditor can obtain certain level of belief from this evidence whether the accounts receivable exist (non-fictitious) or do not exist (fictitious) and also whether the account balance is valued properly or not valued properly. In general, the level of beliefs may differ from one variable to another. For example, in the above case, the auditor may have a high level of belief, say 0.8, that the 'Existence' (e) objective of accounts receivable is met but may have a low level of belief, say 0.6, that the 'Valuation' (v) objective of the account is met[3]. A lower belief for the 'Valuation' objective may be due to the auditor's discovery of some clerical errors in the calculation of the related sales. The above judgment of the auditor can be written in terms of belief functions as:

Bel(e) = 0.8, Bel(~e) = 0,
Bel(v) = 0.6, Bel(~v) = 0,

The question is how should we represent the above beliefs in terms of m-values on the joint space of 'Existence' and 'Valuation'? Shafer, Shenoy, and Srivastava (1988) use the concept of nested beliefs (Shafer 1976) to achieve the above task. However, they did not provide a general solution to the problem, especially, for the cases where you have both positive and negative beliefs on each variable and also where the number of variables involved is bigger than two. Dubois and Prade (1986, 1992, and 1994) have discussed the above issue and shown that one can set-up a Linear Programming problem to find a solution. In the present article we propose an alternative algorithm that provides a solution without the computational effort of solving a linear program.[4] Our algorithm is also supported by a least squares criterion which may be applied to empirical evidence, further encouraging its use in practice. Furthermore, our approach provides m-values for maximally dependent items of evidence (fully dependent items of evidence) which is the situation in auditing.

In the next section of the paper, we describe the algorithm and illustrate its application to a specific example. We follow this section with some concluding remarks.

2. THE ALGORITHM AND AN EXAMPLE

In order to illustrate the algorithm, let us consider a little more complex example than the one described in the introduction. Let us consider that the auditor is evaluating the internal accounting control 'batch totals are compared with computer summary reports for cash receipts'. This evidence bears on three variables: existence, completeness, and valuation of cash receipts (for more examples see Arens et al 2006). In general, the level of support from such items of evidence for each variable may differ. For example, in such a case, the auditor's assessment of the levels of support may be as follows: (1) 0.6 degree of support that the 'existence' objective is met ('e'), and no support for its negation ('~e'), (2) 0.4 degree of support that the 'completeness' objective is met ('c'), and no support that it is not met ('~c'), and (3) 0.3 degree of support that the 'valuation' objective is met ('v') and 0.1 degree of support that it is not met ('~v'). The auditor's judgments can be written[5] in terms of belief functions on each variable as:

Bel(e) = 0.6 and Bel(~e) = 0,
Bel(c) = 0.4 and Bel(~c) = 0,
Bel(v) = 0.3 and Bel(~v) = 0.1.

We will use this example to illustrate an algorithm for the simple assignment of m-values to the frame of discernment.

The Algorithm

Step 1: Express the beliefs in terms of m-values on the individual frames of the variables. For the above example, we will get:
m(e) = 0.6, m(~e) = 0, and m({e,~e}) = 0.4,
m(c) = 0.4, m(~c) = 0, and m({c,~c}) = 0.6,
m(v) = 0.3, m(~v) = 0.1, and m({v,~v}) = 0.6.

Step 2: List the m-values for each variable in a columnar form; columns for variables, and rows for their values (see Table 1).

Step 3: Select the smallest non-zero m-value in each column (i.e., for each variable). These values are written inside highlighted boxes in Table 1. These values define the elements of the joint space.

Step 4: Select the smallest m-value among the set obtained in Step 3. This value represents the m-value for the set of elements on the joint space generated by the product of individual elements corresponding to the m-values selected in Step 3.

Step 5: Subtract the m-value obtained in Step 4 from each selected m-value in Step 3.

Step 6: Repeat Steps 3 - 4 until all entries are zero.

The resulting m-values

The m-values generated on the joint space through the above algorithm for our example are (see Table 1).
m({ec~v, ~ec~v}) = 0.1,
m({ecv, ~ecv}) = 0.3,
m({ecv, ec~v, e~cv, e~c~v, }) = 0.6.

Table 1: Algorithm steps in calculation m-values on the joint space of variables

Step	Variable	m-value	Variable	m-value	Variable	m-value	Result
1	e	0.6	c	$\boxed{0.4}$	v	0.3	
	~e	0.0	~c	0.0	~v	$\boxed{0.1}$	
	{e,~e}	$\boxed{0.4}$	{c,~c}	0.6	{v,~v}	0.6	m({ec~v,~ec~v})=0.1
2	e	0.6	c	$\boxed{0.3}$	v	$\boxed{0.3}$	
	~e	0.0	~c	0.0	~v	0.0	
	{e,~e}	$\boxed{0.3}$	{c,~c}	0.6	{v,~v}	0.6	m({ecv,~ecv})=0.3
3	e	$\boxed{0.6}$	c	0.0	v	0.0	
	~e	0.0	~c	0.0	~v	0.0	
	{e,~e}	0.0	{c,~c}	$\boxed{0.6}$	{v,~v}	$\boxed{0.6}$	m({ecv,ec~v,e~cv,e~c~v})=0.6
4	e	0.0	c	0.0	v	0.0	
	~e	0.0	~c	0.0	~v	0.0	
	{e,~e}	0.0	{c,~c}	0.0	{v,~v}	0.0	Stop

As we can see, the above m-values are not nested. However, for the case of two variables with only positive beliefs, one would obtain nested m-values as used by Shafer, Shenoy, and Srivastava (1988).

If we marginalize the above m-values on the individual variable space then we do get the beliefs that the auditor had estimated. The above approach is valid even for non-binary variables. Of course, m-value assignments with this property are not unique. However, the merit of this particular assignment algorithm may be argued in two ways.

First, the algorithm is computationally economic relative to other approaches such as linear programming. Moreover, it is possible to show that the present algorithm produces the same assignments as linear programming under certain conditions.

Second, we can show that this algorithm produces an assignment of m-values which minimizes the squared differences between each pairwise assignment and the consequent belief value in the case of two variables.

As Dubois and Prade (1986) discuss, the existence of criteria-dependent solutions to the m-value assignment problem is not surprising. However, the computational simplicity of the present algorithm suggests its consideration in practice.

3 MAXIMAL ORDER DECOMPOSITION

The creation of m-values with the algorithm described in the previous section is computationally efficient. In this section, we wish to explore the mathematical properties of the algorithm. It is difficult to develop insights in the general case, so we restrict our attention to the case of two variables. This is similar to the approach taken by Dubois and Prade (1986).

Consider the two variables c and v from the previous section. m-values on their values are easily summarized in :

Table 2: m-values for two variables

Var	v	~v	(v,~v)	m
C	0.3	0.1	0.0	0.4
~c	0.0	0.0	0.0	0.0
(c,~c)	0.0	0.0	0.6	0.6
M	0.3	0.1	0.6	

Many allocations of these m-values are possible consistent with the row and column totals. The allocations in Table 2 from our algorithm can be shown to have some very attractive properties.

For comparative purposes with other assignments, we may calculate two statistics. First, the entropy, Entropy = $-\sum p_i \ln p_i$ and second, the value of the chi-square statistic for testing the hypothesis of independence, $\mathrm{X}^2 = \sum \frac{(O-E)^2}{E}$, where O is the observed table value and E is the table value expected if rows and columns were independent. In the case of independence, table values would be assigned by multiplying row and column marginal totals. For our algorithm,

Entropy=-0.3ln(0.3) -0.1ln(0.1) -0.6ln(0.6) = 0.8979,

$\mathrm{X}^2 = (0.3-0.12)^2/0.12 + ... + (0.6-0.36)^2/0.36 = 1$

If m-values were allocated according to independence, we obtain Entropy = 1.572 and X^2 = 0.0.

The entropy for a joint distribution of two random variables E(X,Y) is known to satisfy E(X,Y) >= E(X), E(X,Y) >= E(Y), and E(X,Y) <= E(X) + E(Y), with the last being an equality if and only if X and Y are independent random variables. In the above table, denote the entropy for the rows and columns by E(C) and E(V). It is easily seen that E(C) = 0.673 and E(V) =0.898. In this example, our algorithm produces a joint entropy equal to that of the columns which, in turn, is the smallest possible joint entropy consistent with the row and column totals. The assignment of m-values via the independence assumption, alternatively, yields a joint entropy that is the highest possible, namely the sum of E(C) and E(V).

Thus our algorithm, when compared with independent allocation, minimizes entropy and maximizes the chi-square statistic. Since entropy is a measure of disorder, we are maximizing order, and hence we term our approach, **Maximal Order Decomposition**. Thus we have a clear distinction with the Dubois and Prade algorithm, which is based on linear programming and maximizes entropy. Thus we have two competing approaches, that of independence, which is equivalent to maximum entropy, and our algorithm, which results in minimum entropy. In our case, the two sets of m-values originate from the same source, so we cannot assume independence. The minimum entropy approach provides m-values for the more realistic fully dependent case.

We believe ours is clearly superior on computational grounds, making it the algorithm of choice in large complex systems. Note also that while independence requires simple multiplication to allocate m-values, the number of nonzero elements in the frame grows exponentially with the number of variables. In the two-variable case, our frame has only three nonzero m-values, while independent variables would have nine. With 25 variables, our approach would yield 25 nonzero m-values, while independent variables would require 3^{25} = 8.5E11 nonzero assignments.

A proof that our approach maximizes chi-square and minimizes entropy is unattainable in the general case, but a proof is available for the simplest 2 x 2 case, which was considered by Dubois and Prade. We will use their notation for ease of comparison. First, consider assigning m-values of α and β to variable a and b. An assignment is defined by X_{AB} , X_A , X_B , X_w , as in the table below:

Table 3: Feasible assignment of m-values

Variable	b	~b	m-value
a	X_{AB}	$\alpha - X_{AB}$	α
~a	$\beta - X_{AB}$	$1-\alpha-\beta+X_{AB}$	$1-\alpha$
m-value	β	$1-\beta$	

The minimum chi-square statistic is zero when $X_{AB}=\alpha * \beta$. The maximum chi-square statistic can be found by maximizing

$$\mathrm{X}^2 = (X_{AB}-\alpha\beta)^2 / (\alpha(1-\alpha)\beta(1-\beta))$$

subject to the constraints

$$\max(0, \alpha + \beta - 1) \le X_{AB} \le \min(\alpha, \beta).$$

Clearly the maximum will occur at either the upper or lower bound on X_{AB}, and we may simply examine all (four) possible orderings of the m-values to verify that our algorithm maximizes chi-square. We will not repeat the proof for all possibilities. The interested reader may find it informative to do so, however. As an example of one of the possibilities, suppose that $\alpha \le \beta \le 1-\beta \le 1-\alpha$. Our algorithm produces the solution corresponding to $X_{AB}=\alpha$. For this particular ordering of m-values, the previously stated limits become $0 \le X_{AB} \le \alpha$. Maximum chi-square occurs at $X_{AB}=\alpha$ if and only if $(\alpha - \alpha\beta)^2 \ge (0 - \alpha\beta)^2$ which is true for this chosen case. Similar arguments hold for any permutation of the m-values, and therefore our **Maximum Order Decomposition Algorithm** maximizes the chi-square statistic for any given set of m-values.

To also prove that the algorithm minimizes entropy, consider any arbitrary allocation as a function of X_{AB}. After writing the expression for Entropy as a function of X_{AB} we note that the same upper and lower bounds as for the chi-square calculation must be preserved. We also note that Entropy is (1) concave downward and (2) has a derivative of zero only at $X_{AB}=\alpha\beta$. This point is therefore a global maximum for E, and is identical to the minimum chi-square point. The minimum entropy therefore must be at either the upper or lower limit on X_{AB} as was the case previously considered. The proof that our algorithm minimizes E proceeds in the same way as before. For each of the (four) possible m-value orderings, we can prove that our assignment minimizes E and coincides with the upper or lower limit on X_{AB}.

Our algorithm therefore maximizes chi-square while minimizing entropy. Because of these properties, we may refer to it as the **Maximal Order Decomposition Algorithm**. We should note also that in the simplest case examined by DuBois and Prade, our assignments are identical to theirs.

4 SUMMARY AND CONCLUSION

We have described a sequential algorithm for the assignment of m-values to subsets of the frame of discernment that are consistent with an overall assignment of beliefs to individual variables. While many such assignments are possible, our algorithm is computationally simple, completely general, and is supported by objective chi-square and entropy criteria. In the simplest case of two variables, this algorithm produces an assignment identical to that of more complicated algorithms.

FOOTNOTES

1. There are several types of audit: the audit of financial statements of a company, compliance audit, income tax audit, operational audit, and assertion audit. In principle, they are all the same; they all involve collection, evaluation, and aggregation of evidence to form an opinion. However, the nature of assertions and the corresponding items of evidence may differ from one type of audit to another. In this article, we use examples from the audit of financial statements. Financial Statements consist of a set of four statements in the USA: balance sheet, income statement, statement of cash flow, and statement of retained earnings. (see, e.g., Arens, Elder, and Beasley 2006, for details on the definitions of various types of audit).

2. In auditing accounts receivable, auditors usually send letters of confirmation to some selected customers of the client to verify the following; (1) whether they owe any money to the company, and (2) the amount they owe is the amount given in the confirmation letter.

3. As a convention, we will use the first letter in the lower case in the name of a variable to represent the values of the variable. For example, for 'Existence', we will use 'e' and '~e', respectively for the two values that the objective is met, and not met.

4. The set of m-values on the joint space that yields the desired beliefs on individual variables is not unique.

5. It should be pointed out that this judgment of the auditor can not be easily represented in terms probabilities.

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